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Low energy excitations of a quasi-2D Bose-Einstein condensate

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Abstract. Starting from the Gross-Pitaevskii energy functional of the 3D Bose-Einstein Condensate, we derive approximately the energy functional and the effective coupling constant of the quasi-2D condensate. The evolution of the quasi-2D condensate wave function is studied by a variational method. Low energy excitation spectra for both positive and negative scattering lengths are analyzed. The condition of collapse instability of a quasi-2D Bose gas with attractive particle interaction is also proposed.

PACS. 03.75.Fi Phase coherent atomic ensembles; quantum condensation phenomena – 03.65.Ge Solutions of wave equations: bound states

Since 1995, when Bose-Einstein Condensation (BEC) in trapped alkali vapors was first observed [1–3], the physics of BEC has been of great interest both for theoretical and experimental physicists. Among others, the influence of dimensionality is now a subject of extensive study. The possibility of tightly confining the motion of trapped particles in one direction and creating a quasi-2D gas has long been suggested theoretically. When the frequency ω_z of the tight confinement is such that $\hbar\omega_z$ is much larger than the thermal energy $k_\mathrm{B}T$ and the meanfield inter-particle interaction $n_0 g$ (n_0 is the gas density, and g the coupling constant), the gas is kinematically 2D, in the confined direction the particles are "frozen" and undergo zero-point oscillations. Recently quasi-2D, as well as 1D, BEC has been observed in MIT [4].

In this paper we will use the variational method based on Gaussian trial wave function [5] to study the low energy collective excitations of a quasi-2D condensate both for positive and negative scattering lengths at $T = 0$. Collective excitations of 3D Bose condensates have been widely investigated theoretically and experimentally and good agreement between the numerical and the experimental results has been found [6]. For quasi-2D and 1D Bose gases with repulsive interactions, analytical expressions for the frequency spectra of the collective excitations have been found by Ho and Ma [7] in the large particle number limit. Our approach is valid for arbitrary particle number and obtains the dependence of the frequency on the number of particles. The paper is nothing more than a useful study of the limiting case of [5] by Pérez-Garcia *et al.* with tightly confinement in one direction.

For 3D dilute Bose gases an appropriate description of the excitations can be obtained from the time-dependent Gross-Pitaevskii (GP) equation for the order parameter [6]

$$\mathrm{i}\hbar\frac{\partial\Psi(\mathbf{r})}{\partial t} = \left[-\frac{\hbar}{2m}\nabla^2 + V_\mathrm{ext}(\mathbf{r}) + g\left|\Psi(\mathbf{r})\right|^2\right]\Psi(\mathbf{r}) \qquad (1)$$

where m is the mass of the particle, $V_\mathrm{ext} = m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)/2$ is the anisotropic harmonic trapping potential, and $g = 4\pi\hbar^2 a/m$ is the coupling constant, where a is the s-wave scattering length. $\Psi(\mathbf{r})$ is normalised to the total number of particles

$$\int \mathrm{d}^3\mathbf{r}\left|\Psi(\mathbf{r})\right|^2 = N. \qquad (2)$$

Equation (1) can also be obtained by a variational procedure [6]

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi = \frac{\delta E}{\delta\Psi^*} \qquad (3)$$

with the energy functional E given by

$$E[\Psi] = \int \mathrm{d}^3\mathbf{r}\left\{\frac{\hbar^2}{2m}\left|\nabla\Psi\right|^2 + V_\mathrm{ext} + \frac{g}{2}\left|\Psi\right|^4\right\}\cdot \qquad (4)$$

For an axially symmetric trap with $\omega_z \gg \omega_0 = \omega_x = \omega_y$ and $\hbar\omega_z \gg n_0 g$, the particles are frozen in the z-direction and undergo zero-point oscillations. Then $\Psi(\mathbf{r})$ can be approximately written as

$$\Psi(\mathbf{r}) = \Phi(\rho)\,\varphi_0(z) \qquad (5)$$

where $\rho = \{x, y\}$ and $\varphi_0 = \left(1/\pi l^2\right)^{1/4}\exp\left(-z^2/2l\right)$ is the normalized ground state wave function of the bare trap

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in z-direction with $l = (\hbar/m\omega_z)^{\frac{1}{2}}$, the width of Gaussian. The normalization condition then reduces to

$$\int \mathrm{d}^2\rho \left|\Phi(\rho)\right|^2 = N. \tag{6}$$

Substituting (5) into (4) and integrating with respect to z, we obtain

$$E[\Phi] = \int \mathrm{d}^2\rho \left\{ \frac{\hbar^2}{2m} \left|\nabla_2 \Phi\right|^2 + V_2 \left|\Phi\right|^2 + \frac{g_2}{2} \left|\Phi\right|^4 \right\} \tag{7}$$

where ∇_2 is the gradient operator in two dimensional space, $V_2 = m\omega_0^2\rho^2/2$ and $g_2 = \sqrt{8\pi}\hbar^2 a/ml$. g_2 is the effective coupling constant of the quasi-2D Bosons. It depends not only on the scattering length a but also on the frequency of the tight confinement through l. The expression of g_2 thus obtained is in accord with that derived from scattering theory in the limit $l \gg |a|$ [8].

Using the variational procedure, one can obtain from equation (7) the quasi-2D time-dependent GP equation

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\Phi = \left[-\frac{\hbar^2}{2m}\nabla_2^2 + V_2(\rho) + g_2 \left|\Phi\right|^2 \right] \Phi. \tag{8}$$

Theoretical analyzes have shown that for 2D Bose gases energy functional and equation of the form (7) and (8) respectively, will suffice for practical purposes [9].

The quasi-2D GP equation (8) can be readily recovered by the minimization of the action L associated to the Lagrangian density $\widetilde{L}$

$$\widetilde{L} = \frac{\mathrm{i}\hbar}{2}\left(\Phi\frac{\partial \Phi^*}{\partial t} - \Phi^*\frac{\partial \Phi}{\partial t} \right) + \frac{\hbar^2}{2m}\left|\nabla_2\Phi\right|^2 + V_2\left|\Phi\right|^2 + \frac{g}{2}\left|\Phi\right|^4. \tag{9}$$

By applying small time-dependent disturbances to the trapping potential $V_2(\rho)$, one expects small oscillations of the order parameter around its ground state value. We will study the evolution of the condensate wave function with use of the variational method based on Ritz's optimization procedure. A Gaussian with some free (time-dependent) parameters is selected as the trial function, since it is precisely the ground state wave function of the noninteracting Bose gases in the harmonic trap, namely

$$\Phi(x,y,t) = A(t) \prod_{\eta=x,y} \exp\left(-\frac{\left[\eta - \eta_0(t)\right]^2}{2w_\eta^2} + \mathrm{i}\eta\alpha_\eta(t) + \mathrm{i}\eta^2\beta_\eta(t) \right). \tag{10}$$

At a given t, this function defines a Gaussian distribution at the position (x_0, y_0) with amplitude $A(t)$, width w_η, slope α_η and curvature β_η where $\eta = \{x, y\}$.

In order to find the equations governing the evolution of these variational parameters, we insert equation (10) into equation (9) and calculate the Lagrangian L by integrating the Lagrangian density over the 2D space coordinates. We find

$$\begin{aligned} L &= \int_{-\infty}^{+\infty} \mathrm{d}^2\rho\widetilde{L} \\ &= N\sum_{\eta=x,y}\left\{ \hbar\left[\eta_0\,\dot{\alpha_\eta} + \left(\eta_0^2 + \frac{w_\eta^2}{2}\right)\dot{\beta_\eta} \right] + \frac{\hbar^2}{2m}\left[(\alpha_\eta + 2\eta_0\beta_\eta)^2 + \frac{1}{2}\left(\frac{1}{w_\eta^2} + 4\beta_\eta^2 w_\eta^2 \right) \right] + \frac{1}{2}m\omega_0^2\left(\eta_0^2 + \frac{1}{2}w_\eta^2 \right) \right\} + \frac{g_2 N^2}{4\pi w_x w_y} \end{aligned} \tag{11}$$

where we have made use of the normalization condition (6), namely $\pi\left|A\right|^2 w_x w_y = N$, to eliminate the parameter A.

Using the Euler-Lagrange equations for each variational parameter, we derive from equation (11) the following equations

$$\ddot{\eta_0} + \omega_0^2\eta_0 = 0 \tag{12}$$

$$\ddot{w_x} + \omega_0^2 w_x = \frac{\hbar^2}{m^2 w_x^3} + \frac{g_2 N}{2\pi m w_x^2 w_y} \tag{13}$$

$$\ddot{w_y} + \omega_0^2 w_y = \frac{\hbar^2}{m^2 w_y^3} + \frac{g_2 N}{2\pi m w_x w_y^2} \tag{14}$$

$$\beta_\eta = \frac{m}{2\hbar}\frac{\dot{w_\eta}}{w_\eta}, \quad \alpha_\eta = \frac{m\,\dot{\eta_0}}{\hbar} - 2\eta_0\beta_\eta \tag{15}$$

where $\eta = \{x, y\}$. Equation (12) shows explicitly that the center of the condensate will oscillate with the bare frequency of the harmonic trap. This fact implies that in the presence of the harmonic trap the motion of the center of mass is exactly decoupled from the internal degrees of freedom, as is generally the case [6]. Equations (13, 14) are nonlinear coupled equations for the width of the condensate, from which frequencies of the low energy excitations can be found. β_η and α_η can be obtained from η_0 and w_η through equation (15). Once $\eta_0(t)$ and $w_\eta(t)$ are worked out, the evolution of the Gaussian-like atomic cloud will be completely determined.

With the use of suitably scaled variables $\tau = \omega_0 t$, $\widetilde{g_2} = g_2 m/2\pi\hbar^2$ and $\widetilde{w_\eta} = w_\eta/a_\perp$ where $a_\perp = \sqrt{\hbar/m\omega_0}$, equations (13, 14) can be written in simple form as follows.

$$\frac{\mathrm{d}^2\widetilde{w_x}}{\mathrm{d}\tau^2} + \widetilde{w_x} = \frac{1}{\widetilde{w_x}^3} + \frac{\widetilde{g_2}N}{\widetilde{w_x}^2\widetilde{w_y}}, \tag{16}$$

$$\frac{\mathrm{d}^2\widetilde{w_y}}{\mathrm{d}\tau^2} + \widetilde{w_y} = \frac{1}{\widetilde{w_y}^3} + \frac{\widetilde{g_2}N}{\widetilde{w_x}\widetilde{w_y}^2}\cdot \tag{17}$$

Equations (16, 17) describe small oscillations around the equilibrium width, which corresponds to the stationary solution of the equations. Taking into account the symmetry of the system, the width of equilibrium can be readily obtained

$$\widetilde{w_0} = \widetilde{w_x} = \widetilde{w_y} = (1 + \widetilde{g_2}N)^{\frac{1}{4}}. \tag{18}$$

Equation (18) gives explicitly the dependence of the equilibrium width on the effective coupling constant and particle number. For attractive particle interactions $\widetilde{w_0}$ will no longer be a real number when $|\widetilde{g_2}N| > 1$, which implies a collapse of the condensate.

Expanding equations (16, 17) around the equilibrium point defined by (18) to the first power of $\delta\widetilde{w_\eta} = \widetilde{w_\eta} - \widetilde{w_0}$, we obtain equations for $\delta\widetilde{w_\eta}$

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\delta\widetilde{w_x} + \delta\widetilde{w_x} = -\frac{3}{\widetilde{w_0}^4}\delta\widetilde{w_x} - \frac{2\widetilde{g_2}N}{\widetilde{w_0}^4}\delta\widetilde{w_x} - \frac{\widetilde{g_2}N}{\widetilde{w_0}^4}\delta\widetilde{w_y} \tag{19}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\delta\widetilde{w_y} + \delta\widetilde{w_y} = -\frac{3}{\widetilde{w_0}^4}\delta\widetilde{w_y} - \frac{2\widetilde{g_2}N}{\widetilde{w_0}^4}\delta\widetilde{w_y} - \frac{\widetilde{g_2}N}{\widetilde{w_0}^4}\delta\widetilde{w_x}. \tag{20}$$

Introducing $\delta\widetilde{w_\pm} = \delta\widetilde{w_x} \pm \delta\widetilde{w_y}$, the above equations can be converted into a simple decoupled form

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\delta\widetilde{w_+} + 4\delta\widetilde{w_+} = 0, \tag{21}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\delta\widetilde{w_-} + \frac{4 + 2\widetilde{g_2}N}{1 + \widetilde{g_2}N}\delta\widetilde{w_-} = 0. \tag{22}$$

Equations (21, 22) show that there appear two oscillatory modes in the condensate with frequencies

$$\omega_a = 2\omega_0 \tag{23}$$

and

$$\omega_b = \omega_0\sqrt{\frac{4 + 2\widetilde{g_2}N}{1 + \widetilde{g_2}N}} \tag{24}$$

ω_a corresponds to a breathing oscillation. The existence of this breathing oscillatory mode with frequency $2\omega_0$ is a direct consequence of the hidden symmetry of the system. For 2D atoms in a harmonic trap interacting by a local potential, one expects well-defined modes with a frequency of exactly $2\omega_0$ [10]. ω_b depends on $\widetilde{g_2}$ as well as on N. In the noninteracting limit $\widetilde{g_2}N \to 0$, one has $\omega_a = \omega_b = 2\omega_0$, as predicted for the ideal oscillator. For a system with a large number of particles $\widetilde{g_2}N \gg 1$, one has $\omega_b = \sqrt{2}\omega_0$, in agreement with the result of Ho and Ma for $n = |m| = 2$ [7]. The dependence of ω_b on N is plotted in Figure 1.

To understand the nature of these excitations, we compare them with the results of Ho and Ma [7] and classify the modes *via* their rotational properties. With m as the quantum number of the angular momentum along z-axis, ω_a is referred to as $m = 0$ mode, the center of mass motion as $m = +1$ and $m = -1$ modes, and ω_b corresponds to a linear superposition of $m = +2$ and $m = -2$ modes with real and equal-absolute-value coefficients.

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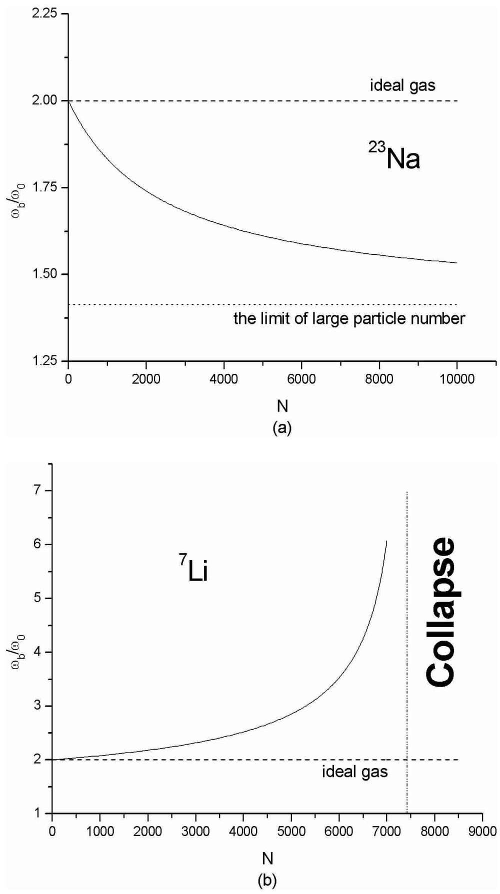

Fig. 1. Oscillation frequency ω_b as a function of the particle number N of the condensate for positive (a) and negative (b) scattering length, taking parameters of ^{23}Na and ^{7}Li respectively and $\omega_z/2\pi = 790$ Hz. The limits of ideal gas (the dashed line) and large particle number (the dotted line) are also shown.

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